

Analytical Blowup Solutions to the Isothermal Euler-Poisson Equations of Gaseous Stars in R^N

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Abstract

This article is the continued version of the analytical blowup solutions for 2-dimensional Euler-Poisson equations in [10] and [11]. With the extension of the blowup solutions with radial symmetry for the isothermal Euler-Poisson equations in R^2 , other special blowup solutions in R^N with non-radial symmetry are constructed by the separation method.

Key words: Analytical Solutions, Euler-Poisson Equations, Isothermal, Blowup, Special Solutions, Non-Radial Symmetry

1 Introduction

The evolution of a self-gravitating fluid (gaseous stars) can be formulated by the isentropic Euler-Poisson equations of the following form:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0, \\ (\rho \vec{u})_t + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla P = -\rho \nabla \Phi, \\ \Delta \Phi(t, \vec{x}) = \alpha(N)\rho, \end{cases} \quad (1)$$

where $\alpha(N)$ is a constant related to the unit ball in R^N : $\alpha(1) = 2$; $\alpha(2) = 2\pi$ and For $N \geq 3$,

$$\alpha(N) = N(N-2)V(N) = N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2+1)}, \quad (2)$$

where $V(N)$ is the volume of the unit ball in R^N and Γ is a Gamma function. And as usual, $\rho = \rho(t, \vec{x})$ and $\vec{u} = \vec{u}(t, \vec{x}) = (u_1, u_2, \dots, u_N) \in \mathbf{R}^N$ are the density, the velocity respectively. $P = P(\rho)$ is the pressure.

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In the above system, the self-gravitational potential field $\Phi = \Phi(t, \vec{x})$ is determined by the density ρ through the Poisson equation.

The equation (1)₃ is the Poisson equation through which the gravitational potential is determined by the density distribution of the density itself. Thus, we call the system (1) the Euler-Poisson equations. The equations can be viewed as a perfect gas model. The function $P = P(\rho)$ is the pressure. The γ -law can be applied on the pressure $P(\rho)$, i.e.

$$P(\rho) = K\rho^\gamma := \frac{\rho^\gamma}{\gamma}, \quad (3)$$

which is a commonly the hypothesis. The constant $\gamma = c_P/c_v \geq 1$, where c_P, c_v are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in (3). In particular, the fluid is called isothermal if $\gamma = 1$. It can be used for constructing models with non-degenerate isothermal cores, which have a role in connection with the so-called Schonberg-Chandrasekhar limit [7]. And we denote the radial diameter as: $r := \sqrt{\sum_{j=1}^N x_j^2}$.

The system can be rewritten as

$$\begin{cases} \rho_t + \nabla \cdot \vec{u}\rho + \nabla \rho \cdot \vec{u} = 0, \\ \rho \left(\frac{\partial u_i}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial}{\partial x_i} P(\rho) = -\rho \frac{\partial}{\partial x_i} \Phi(\rho), \text{ for } i = 1, 2, \dots, N, \\ \Delta \Phi(t, x) = \alpha(N)\rho. \end{cases} \quad (4)$$

For $N = 3$, (4) is a classical (non-relativistic) description of a galaxy, in astrophysics. See [2], [3] and [7] for a detail about the system.

For the local existence results about the system were shown in [1] and [5]. Historically in astrophysics, Goldreich and Weber constructed the analytical blowup (collapsing) solutions of the 3-dimensional Euler-Poisson equations for $\gamma = 4/3$ for the non-rotating gas spheres [6]. After that, Makino [8] obtained the rigorously mathematical proof of the existence of such kind of blowup solutions. Besides, Deng, Xiang and Yang extended the above blowup solutions in R^N ($N \geq 4$) [4]. Recently, Yuen obtained the blowup solutions in R^2 with $\gamma = 1$ by a new transformation [10]. The family of the analytical solutions with radial symmetry

$$\vec{u} = \frac{\vec{x}}{|r|} V(t, r),$$

are rewritten as

For $N \geq 3$ and $\gamma = (2N - 2)/N$, in [9]

$$\begin{cases} \rho(t, r) = \begin{cases} \frac{1}{a^N(t)} y \left(\frac{r}{a(t)} \right)^{N/(N-2)}, & \text{for } r < a(t)Z_\mu; \\ 0, & \text{for } a(t)Z_\mu \leq r. \end{cases}, \quad V(t, r) = \frac{\dot{a}(t)}{a(t)} r, \\ \ddot{a}(t) = \frac{-\lambda}{a^{N-1}(t)}, \quad a(0) = a_1 \neq 0, \quad \dot{a}(0) = a_2, \\ \ddot{y}(z) + \frac{N-1}{z} \dot{y}(z) + \frac{\alpha(N)}{(2N-2)K} y(z)^{N/(N-2)} = \mu, \quad y(0) = \alpha > 0, \quad \dot{y}(0) = 0, \end{cases} \quad (5)$$

where $\mu = [N(N-2)\lambda]/(2N-2)K$ and the finite Z_μ is the first zero of $y(z)$;

For $N = 2$ and $\gamma = 1$, in [10]

$$\left\{ \begin{array}{l} \rho(t, r) = \frac{1}{a^2(t)} e^{y(r/a(t))}, \quad V(t, r) = \frac{\dot{a}(t)}{a(t)} r; \\ \ddot{a}(t) = \frac{-\lambda}{a(t)}, \quad a(0) = a_1 > 0, \quad \dot{a}(0) = a_2; \\ \ddot{y}(z) + \frac{1}{z} \dot{y}(z) + \frac{\alpha(2)}{K} e^{y(z)} = \mu, \quad y(0) = \alpha, \quad \dot{y}(0) = 0, \end{array} \right. \quad (6)$$

where $K > 0$, $\mu = 2\lambda/K$ with a sufficiently small λ and α are constants.

However, all the known solutions are in radial symmetry. In this paper, we are able to obtain the similar results to the non-radial symmetric cases for the 2-dimensional Euler-Poisson equations (4) in the following theorem.

Theorem 1 *For the isothermal Euler-Poisson equations (4) in R^N , there exists a family of solutions,*

$$\left\{ \begin{array}{l} \rho(t, \vec{x}) = \frac{1}{a(t)} e^{-\frac{\Phi\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{K} + C}, \quad \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{2a(t)} (x_1, x_2, x_1 + x_2, \dots, x_1 + x_2), \\ a(t) = \frac{1}{4a_1} (a_2 t + 2a_1)^2 \\ s\ddot{\Phi}(s) + \frac{\dot{\Phi}(s)}{2} - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \quad \Phi(0) = \alpha, \quad \dot{\Phi}(0) = \epsilon^* e^{-\frac{\alpha}{K}}, \end{array} \right. \quad (7)$$

where $A, B, a_1 \neq 0, a_2, \frac{\alpha(N)e^C}{4} = \epsilon^* > 0, \alpha$ and β are constants.

In particular, $a_1 > 0$ and $a_2 < 0$, the solutions (7) blow up in the finite time $T = -a_2/a_1$.

2 Special Blowup Solutions I

Before presenting the proof of Theorem 1, we prepare the following two lemmas.

Lemma 2 *For the continuity equation (4)₁ in R^N , there exist solutions,*

$$\rho(t, \vec{x}) = \frac{f\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a(t)}, \quad \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{2a(t)} (x_1, x_2, x_1 + x_2, \dots, x_1 + x_2), \quad (8)$$

where the scalar function $f(s) \geq 0 \in C^1$ and $a(t) \neq 0 \in C^1$.

Proof. We plug the solutions (7) into the continuity equation (4)₁,

$$\rho_t + \nabla \cdot \vec{u}\rho + \nabla \rho \cdot \vec{u} \quad (9)$$

$$= \frac{\partial}{\partial t} \left[\frac{f\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a(t)} \right] + \nabla \cdot \frac{\dot{a}(t)}{2a(t)}(x_1, x_2, x_1 + x_2, \dots, x_1 + x_2) \frac{f\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a(t)} \quad (10)$$

$$+ \nabla \frac{f\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a(t)} \cdot \frac{\dot{a}(t)}{a(t)}(x_1, x_1, x_3 + x_2, \dots, x_2 + x_2) \quad (11)$$

$$= \frac{-\dot{a}(t)}{a^2(t)} f\left(\frac{x_1^2 + x_2^2}{a(t)}\right) + \frac{1}{a(t)} \frac{\partial}{\partial t} f\left(\frac{x_1^2 + x_2^2}{a(t)}\right) + \frac{\dot{a}(t)}{2a(t)} \left(\frac{\partial}{\partial x_1} x_1 + \frac{\partial}{\partial x_2} x_2 + \sum_{i=3}^N \frac{\partial}{\partial x_i} (x_1 + x_2) \right) \frac{f\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a(t)} \quad (12)$$

$$+ \frac{\dot{a}(t)}{2a(t)} \left[\frac{\partial}{\partial x_1} \frac{f\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a(t)} \cdot x_1 + \frac{\partial}{\partial x_2} \frac{f\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a(t)} \cdot x_2 + \sum_{i=3}^N \frac{\partial}{\partial x_i} \frac{f\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a(t)} \cdot (x_1 + x_2) \right] \quad (13)$$

$$= \frac{-\dot{a}(t)}{a^2(t)} f\left(\frac{x_1^2 + x_2^2}{a(t)}\right) - \frac{1}{a(t)} \dot{f}\left(\frac{x_1^2 + x_2^2}{a(t)}\right) \frac{(x_1^2 + x_2^2)\dot{a}(t)}{a^2(t)} + \frac{\dot{a}(t)}{a(t)} \frac{f\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a^2(t)} \quad (14)$$

$$+ \frac{\dot{a}(t)}{2a(t)} \left[\frac{\dot{f}\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a^2(t)} \frac{2x_1^2}{a(t)} + \frac{\dot{f}\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{a^2(t)} \frac{2x_2^2}{a(t)} \right] \quad (15)$$

$$= 0. \quad (16)$$

The proof is completed. ■

The following lemma handles the Poisson equation (4)₃ for our solutions (7):

Lemma 3 *The solutions,*

$$\rho = \frac{1}{a(t)} e^{-\frac{\Phi\left(\frac{x_1^2 + x_2^2}{a(t)}\right)}{K}} + C, \quad (17)$$

with the second-order ordinary differential equation:

$$s\ddot{\Phi}(s) + \frac{\dot{\Phi}(s)}{2} - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \quad \Phi(0) = \alpha, \quad \dot{\Phi}(0) = \epsilon^* e^{-\frac{\alpha}{K}}, \quad (18)$$

where $s := (x_1^2 + x_2^2)/a(t)$ and $C, \frac{\alpha(N)e^C}{4} = \epsilon^*$ and α are constants,

fit into the Poisson equation (4)₃ in R^N .

Proof. We check that our potential function $\Phi(t, x_1, x_2)$ satisfies the Poisson equation (4)₃:

$$\Delta\Phi(t, \vec{x}) - \alpha(N)\rho \quad (19)$$

$$= \nabla \cdot \nabla \Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right) - \frac{\alpha(N)}{a(t)} e^{-\frac{\Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right)}{K} + C} \quad (20)$$

$$= \nabla \cdot \left[\frac{\partial}{\partial x_1} \Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right), \frac{\partial}{\partial x_2} \Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right), \frac{\partial}{\partial x_3} \Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right), \dots, \frac{\partial}{\partial x_N} \Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right) \right] \quad (21)$$

$$- \frac{\alpha(N)}{a(t)} e^{-\frac{\Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right)}{K} + C} \quad (22)$$

$$= \nabla \cdot \left[\dot{\Phi} \left(\frac{x_1^2 + x_2^2}{a(t)} \right) \frac{2x_1}{a(t)}, \dot{\Phi} \left(\frac{x_1^2 + x_2^2}{a(t)} \right) \frac{2x_2}{a(t)}, 0 \right] - \frac{\alpha(N)}{a(t)} e^{-\frac{\Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right)}{K} + C} \quad (23)$$

$$= \frac{\partial}{\partial x_1} \left[\dot{\Phi} \left(\frac{x_1^2 + x_2^2}{a(t)} \right) \frac{2x_1}{a(t)} \right] + \frac{\partial}{\partial x_2} \left[\dot{\Phi} \left(\frac{x_1^2 + x_2^2}{a(t)} \right) \frac{2x_2}{a(t)} \right] - \frac{\alpha(N)}{a^2(t)} e^{-\frac{\Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right)}{K} + C} \quad (24)$$

$$= \ddot{\Phi} \left(\frac{x_1^2 + x_2^2}{a(t)} \right) \frac{4x_1^2}{a^2(t)} + \dot{\Phi} \left(\frac{x_1^2 + x_2^2}{a(t)} \right) \frac{2}{a(t)} \quad (25)$$

$$+ \ddot{\Phi} \left(\frac{x_1^2 + x_2^2}{a(t)} \right) \frac{4x_2^2}{a^2(t)} + \dot{\Phi} \left(\frac{x_1^2 + x_2^2}{a(t)} \right) \frac{2}{a(t)} - \frac{\alpha(N)}{a(t)} e^{-\frac{\Phi \left(\frac{x_1^2 + x_2^2}{a(t)} \right)}{K} + C} \quad (26)$$

$$= \frac{4}{a(t)} \left(s\ddot{\Phi}(s) + \frac{\dot{\Phi}(s)}{2} - \frac{\alpha(N)e^C}{4} e^{-\frac{\Phi(s)}{K}} \right), \quad (27)$$

where we choose $s := (x_1^2 + x_2^2)/a(t)$ and the ordinary differential equation:

$$s\ddot{\Phi}(s) + \frac{\dot{\Phi}(s)}{2} - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \quad \Phi(0) = \alpha, \quad \dot{\Phi}(0) = \beta, \quad (28)$$

with $\frac{\alpha(N)e^C}{4} = \epsilon^*$, α and β are constants. Therefore, our solutions (71) satisfy the Poisson equation (4)₃.

The proof is completed. ■

Besides, we need the lemma for stating the property of the function $\Phi(s)$ of the analytical solutions (18). We need the lemma for stating the property of the function $\Phi(s)$. In particular, the solutions (7) in N -dimensional case involve the following lemma. The similar lemma was already given in Lemmas 9 and 10, in [10], by the fixed point theorem. For the completeness of understanding the whole article, the proof is also presented here.

Lemma 4 *There exists a sufficiently small $\epsilon^* > 0$, such that the ordinary differential equation*

$$\begin{cases} s\ddot{\Phi}(s) + \frac{\dot{\Phi}(s)}{2} - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \\ \Phi(0) = \alpha, \quad \dot{\Phi}(0) = -\epsilon^* e^{-\frac{\alpha}{K}} \end{cases} \quad (29)$$

where $K > 0$ and α are constants, has a unique solution $\Phi(s) \in C^2[0, \infty)$.

Proof. The lemma can be proved by the fixed point theorem. The equation (29) can be rewritten as:

$$s^{1/2} \frac{d}{ds} \left(s^{1/2} \dot{\Phi}(s) \right) = -\varepsilon^* e^{-\frac{\Phi(s)}{K}}, \quad (30)$$

$$\frac{d}{ds} \left(s^{1/2} \dot{\Phi}(s) \right) = \frac{-\varepsilon^* e^{-\frac{\Phi(s)}{K}}}{s^{1/2}}. \quad (31)$$

With the initial conditions: $\Phi(0) = \alpha$ and $\dot{\Phi}(0) = -\varepsilon^* e^{-\frac{\alpha}{K}}$, the equation (29) is reduced to

$$\dot{\Phi}(s) = \frac{-\varepsilon^*}{s^{1/2}} \int_0^s \frac{e^{-\frac{\Phi(\tau)}{K}}}{\tau^{1/2}} d\tau. \quad (32)$$

Set

$$f(s, \Phi(s)) = \frac{-\varepsilon^*}{s^{1/2}} \int_0^s \frac{e^{-\frac{\Phi(\tau)}{K}}}{\tau^{1/2}} d\tau. \quad (33)$$

For any $s_0 > 0$, we get $f \in C^1[0, s_0]$. And for any $\Phi_1, \Phi_2 \in C^2[0, s_0]$, we have,

$$|f(s, \Phi_1(s)) - f(s, \Phi_2(s))| = \frac{\varepsilon^* \left| \int_0^s \left(\frac{e^{-\frac{\Phi_2(\tau)}{K}} - e^{-\frac{\Phi_1(\tau)}{K}}}{\tau^{1/2}} \right) d\tau \right|}{s^{1/2}}. \quad (34)$$

As e^Φ is a C^1 function of Φ , we can show that the function e^Φ , is Lipschitz-continuous. Then we get,

$$|f(s, \Phi_1(s)) - f(s, \Phi_2(s))| = \frac{\varepsilon^* \int_0^s \left| \frac{(\Phi_2(\tau) - \Phi_1(\tau))}{\tau^{1/2}} \right| d\tau}{K s^{1/2}} \leq \frac{\varepsilon^*}{K} \sup_{0 \leq s \leq s_0} |\Phi_1(\tau) - \Phi_2(\tau)|. \quad (35)$$

Let

$$T\Phi(s) = \alpha + \int_0^s f(\tau, \Phi(\tau)) d\tau. \quad (36)$$

We have $T\Phi \in C[0, s_0]$ and

$$|T\Phi_1(s) - T\Phi_2(s)| = \left| \int_0^s f(\tau, \Phi_1(\tau)) d\tau - \int_0^s f(\tau, \Phi_2(\tau)) d\tau \right| \leq \frac{\varepsilon^*}{K} \sup_{0 \leq s \leq s_0} |\Phi(s)_1 - \Phi(s)_2|. \quad (37)$$

By choosing the constant ε^* such that $0 < \frac{\varepsilon^*}{K} < 1$, this shows that the mapping $T : C[0, s_0] \rightarrow C[0, s_0]$, is a contraction with the sup-norm. By the fixed point theorem, there exists a unique $\Phi(s) \in C[0, s_0]$, such that $T\Phi(s) = \Phi(s)$.

It is because that the chosen constant ε^* is independent of the variable x . Therefore, we have the global unique solution $\Phi(s) \in C^2[0, \infty)$. The proof is completed. ■

Now, we are ready to check that the solutions fit into the Euler-Poisson equations (4).

Proof of Theorem 1. By Lemma 7 and Lemma 8, the solutions (7) satisfy (4)₁ and (4)₃. For the x_i -component of the isothermal momentum equations (4)₃ in R^N ($N \geq 3$), we have

$$\rho \left(\frac{\partial u_1}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_1}{\partial x_k} \right) + \frac{\partial}{\partial x} K \rho + \rho \frac{\partial \Phi}{\partial x} \quad (38)$$

$$= \rho \left[\frac{\partial}{\partial t} \frac{\dot{a}(t)}{2a(t)} x_1 + \frac{\dot{a}(t)x_1}{2a(t)} \frac{\partial}{\partial x_1} \frac{\dot{a}(t)}{2a(t)} + \frac{\dot{a}(t)x_2}{2a(t)} \frac{\partial}{\partial x_2} \frac{\dot{a}(t)x_1}{2a(t)} \right. \\ \left. + \sum_{i=3}^N \frac{\dot{a}(t)(x_1+x_2)}{2a(t)} \frac{\partial}{\partial x_i} \left(\frac{\dot{a}(t)x_1}{2a(t)} \right) \right] \quad (39)$$

$$+ K \frac{\partial}{\partial x} \frac{e^{-\frac{\Phi\left(\frac{x_1^2+x_2^2}{a(t)}\right)}{K}+C}}{a(t)} + \rho \Phi \left(\frac{x_1^2+x_2^2}{a(t)} \right) \frac{A}{a(t)} \quad (40)$$

$$= \rho \left[\frac{1}{2} \left(\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) x_1 + \frac{1}{4} \frac{\dot{a}(t)x_1}{a(t)} \frac{\dot{a}(t)}{a(t)} \right] \quad (41)$$

$$- K \frac{e^{-\frac{\Phi\left(\frac{x_1^2+x_2^2}{a(t)}\right)}{K}+C}}{a^2(t)} \Phi \left(\frac{x_1^2+x_2^2}{a(t)} \right) \frac{2x_1}{Ka(t)} + \rho \Phi \left(\frac{x_1^2+x_2^2}{a(t)} \right) \frac{2x_1}{a(t)} \quad (42)$$

$$= \rho \left[\frac{1}{4} \left(\frac{2\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) x_1 \right] - \rho \Phi \left(\frac{x_1^2+x_2^2}{a(t)} \right) \frac{2x_1}{a(t)} + \rho \Phi \left(\frac{x_1^2+x_2^2}{a(t)} \right) \frac{2x_1}{a(t)} \quad (43)$$

$$= 0, \quad (44)$$

where we used

$$2a(t)\ddot{a}(t) - \dot{a}^2(t) = 0, \quad a(0) = a_1 > 0, \quad \dot{a}(0) = a_2. \quad (45)$$

which is exactly solvable by Maple,

$$a(t) = \frac{1}{4a_1} (a_2 t + 2a_1)^2. \quad (46)$$

For the x_2 -component of the isothermal momentum equations (4)₃ in R^N , we have

$$\rho \left(\frac{\partial u_2}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_2}{\partial x_k} \right) + \frac{\partial}{\partial x_2} K \rho + \rho \frac{\partial \Phi}{\partial x_2} \quad (47)$$

$$= \rho \left[\frac{\partial}{\partial t} \frac{\dot{a}(t)x_2}{2a(t)} + \frac{\dot{a}(t)x_1}{2a(t)} \frac{\partial}{\partial x_1} \frac{\dot{a}(t)}{2a(t)} x_2 + \frac{\dot{a}(t)}{2a(t)} x_2 \frac{\partial}{\partial x_2} \frac{\dot{a}(t)x_2}{2a(t)} + \sum_{k=3}^N u_k \frac{\partial}{\partial x_k} \frac{\dot{a}(t)}{2a(t)} (x_1 + x_2) \right] \quad (48)$$

$$+ K \frac{\partial}{\partial x} \frac{e^{-\frac{\Phi\left(\frac{x_1^2+x_2^2}{a(t)}\right)}{K}+C}}{a^2(t)} + \rho \frac{\partial \Phi \left(\frac{x_1^2+x_2^2}{a(t)} \right)}{\partial x_2} \quad (49)$$

$$= \rho \left[\frac{1}{2} \left(\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) x_2 + \frac{\dot{a}(t)x_2}{4a(t)} \frac{\dot{a}(t)}{a(t)} \right] + 0 + 0 \quad (50)$$

$$= \rho \left[\frac{1}{4} \left(\frac{2\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) x_2 \right] \quad (51)$$

$$= 0. \quad (52)$$

For the x_i -component ($i \geq 3$) of the isothermal momentum equations (4)₃ in R^N , we have

$$\rho \left(\frac{\partial u_i}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial}{\partial x_i} K \rho + \rho \frac{\partial \Phi}{\partial x_i} \quad (53)$$

$$= \rho \left[\frac{\partial}{\partial t} \frac{\dot{a}(t)(x_1+x_2)}{2a(t)} + \frac{\dot{a}(t)x_1}{2a(t)} \frac{\partial}{\partial x_1} \frac{\dot{a}(t)x_1}{2a(t)} + \frac{\dot{a}(t)x_2}{2a(t)} \frac{\partial}{\partial x_2} \frac{\dot{a}(t)x_2}{2a(t)} \right. \\ \left. + \sum_{k=3}^N u_k \frac{\partial}{\partial x_k} \frac{\dot{a}(t)(x_1+x_2)}{2a(t)} \right] \quad (54)$$

$$+ K \frac{\partial}{\partial x_i} \frac{e^{-\frac{\Phi\left(\frac{x_1^2+x_2^2}{a(t)}\right)+C}}{a^2(t)} + \rho \frac{\partial}{\partial x_i} \dot{\Phi} \left(\frac{x_1^2+x_2^2}{a(t)} \right) \quad (55)$$

$$= \rho \left[\frac{1}{2} \left(\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) (x_1+x_2) + \frac{\dot{a}(t)x_1}{2a(t)} \frac{\dot{a}(t)}{2a(t)} + \frac{\dot{a}(t)x_2}{2a(t)} \frac{\dot{a}(t)}{2a(t)} \right] \quad (56)$$

$$= \rho \left[\frac{1}{4} \left(\frac{2\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) (x_1+x_2) \right] \quad (57)$$

$$= 0. \quad (58)$$

Therefore, our solutions satisfy the Euler-Poisson equations. In particular, $a_1 > 0$ and $a_2 < 0$, the solutions (7) blow up in the finite time $T = -a_2/a_1$.

The proof is completed. ■

It is clear to see the blowup rate of the solutions (7):

Corollary 5 *The blowup rate of the solutions (7) is,*

$$\lim_{t \rightarrow T} \rho(t, \vec{0}) (T-t)^2 \geq O(1). \quad (59)$$

3 Special Blowup Solutions II

In the recent paper [11], we have the special solutions for the isothermal Euler-Poisson equations in R^2 , in the following form:

$$\begin{cases} \rho(t, x_1, x_2) = \frac{1}{a^2(t)} e^{-\frac{\Phi\left(\frac{Ax_1+Bx_2}{a(t)}\right)+C}} , \vec{u}(t, x_1, x_2) = \frac{\dot{a}(t)}{a(t)} (x_1, x_2), \\ a(t) = a_1 + a_2 t, \\ \ddot{\Phi}(s) - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \Phi(0) = \alpha, \dot{\Phi}(0) = \beta, \end{cases} \quad (60)$$

where $A, B, a_1 \neq 0, a_2, \frac{2\pi e^C}{A^2+B^2} = \epsilon^* > 0, \alpha$ and β are constants.

In this section, we extend the above solutions to the N -dimensional Euler-Poisson equations (4) in the following theorem:

Theorem 6 *For the isothermal Euler-Poisson equations (4) in R^N ($N \geq 3$), there exists a family*

of solutions,

$$\begin{cases} \rho(t, \vec{x}) = \frac{1}{a^2(t)} e^{-\frac{\Phi\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{K} + C}, \quad \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{a(t)}(x_1, x_2, x_1, \dots, x_1), \\ a(t) = a_1 + a_2 t, \\ \ddot{\Phi}(s) - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \quad \Phi(0) = \alpha, \quad \dot{\Phi}(0) = \beta, \end{cases} \quad (61)$$

where $A, B, a_1 \neq 0, a_2, \frac{\alpha(N)e^C}{A^2+B^2} = \epsilon^* > 0, \alpha$ and β are constants.

In particular, $a_1 > 0$ and $a_2 < 0$, the solutions (61) blow up in the finite time $T = -a_2/a_1$.

Before presenting the proof of Theorem 1, we prepare the following two lemmas.

Lemma 7 For the continuity equation $(4)_1$ in R^N , there exist solutions,

$$\rho(t, \vec{x}) = \frac{f\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)}, \quad \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{a(t)}(x_1, x_2, x_1, \dots, x_1), \quad (62)$$

where the scalar function $f(s) \geq 0 \in C^1$ and $a(t) \neq 0 \in C^1$.

Proof. We plug the solutions (61) into the continuity equation $(4)_1$,

$$\rho_t + \nabla \cdot \vec{u}\rho + \nabla \rho \cdot \vec{u} \quad (63)$$

$$= \frac{\partial}{\partial t} \left[\frac{f\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \right] + \nabla \cdot \frac{\dot{a}(t)}{a(t)}(x_1, x_2, x_1, \dots, x_1) \frac{f\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \quad (64)$$

$$+ \nabla \frac{f\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \cdot \frac{\dot{a}(t)}{a(t)}(x_1, x_2, x_1, \dots, x_1) \quad (65)$$

$$= \frac{-2\dot{a}(t)}{a^3(t)} f\left(\frac{Ax_1+Bx_2}{a(t)}\right) + \frac{1}{a^2(t)} \frac{\partial}{\partial t} f\left(\frac{Ax_1+Bx_2}{a(t)}\right) + \frac{\dot{a}(t)}{a(t)} \left(\frac{\partial}{\partial x_1} x_1 + \frac{\partial}{\partial x_2} x_2 + \sum_{i=3}^N \frac{\partial}{\partial x_i} x_i \right) \frac{f\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \quad (66)$$

$$+ \frac{\dot{a}(t)}{a(t)} \left[\frac{\partial}{\partial x_1} \frac{f\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \cdot x_1 + \frac{\partial}{\partial x_2} \frac{f\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \cdot x_2 + \sum_{i=3}^N \frac{\partial}{\partial x_i} \frac{f\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \cdot x_i \right] \quad (67)$$

$$= \frac{-2\dot{a}(t)}{a^3(t)} f\left(\frac{Ax_1+Bx_2}{a(t)}\right) - \frac{1}{a^2(t)} \dot{f}\left(\frac{Ax_1+Bx_2}{a(t)}\right) \frac{(Ax_1+Bx_2)\dot{a}(t)}{a^2(t)} + 2 \frac{\dot{a}(t)}{a(t)} \frac{f\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \quad (68)$$

$$+ \frac{\dot{a}(t)}{a(t)} \left[\frac{\dot{f}\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \frac{Ax_1}{a(t)} + \frac{\dot{f}\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{a^2(t)} \frac{Bx_2}{a(t)} \right] \quad (69)$$

$$= 0. \quad (70)$$

The proof is completed. ■

The following lemma handles the Poisson equation (4)₃ for our solutions (61):

Lemma 8 *The solutions,*

$$\rho = \frac{1}{a^2(t)} e^{-\frac{\Phi\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{K} + C}, \quad (71)$$

with the second-order ordinary differential equation:

$$\ddot{\Phi}(s) - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \quad \Phi(0) = \alpha, \quad \dot{\Phi}(0) = \beta, \quad (72)$$

where $s := (Ax_1 + Bx_2)/a(t)$ and $C, \frac{\alpha(N)e^C}{A^2+B^2} = \epsilon^*$, α and β are constants, fit into the Poisson equation (4)₃ in R^N .

Proof. We check that our potential function $\Phi(t, x_1, x_2)$ satisfies the Poisson equation (4)₃:

$$\Delta\Phi(t, x_1, x_2) - \alpha(N)\rho \quad (73)$$

$$= \nabla \cdot \nabla \Phi \left(\frac{Ax_1 + Bx_2}{a(t)} \right) - \frac{\alpha(N)}{a^2(t)} e^{-\frac{\Phi\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{K} + C} \quad (74)$$

$$= \nabla \cdot \left[\frac{\partial}{\partial x_1} \Phi \left(\frac{Ax_1 + Bx_2}{a(t)} \right), \frac{\partial}{\partial x_2} \Phi \left(\frac{Ax_1 + Bx_2}{a(t)} \right), \frac{\partial}{\partial x_3} \Phi \left(\frac{Ax_1 + Bx_2}{a(t)} \right), \dots, \frac{\partial}{\partial x_N} \Phi \left(\frac{Ax_1 + Bx_2}{a(t)} \right) \right] \quad (75)$$

$$- \frac{\alpha(N)}{a^2(t)} e^{-\frac{\Phi\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{K} + C} \quad (76)$$

$$= \nabla \cdot \left[\dot{\Phi} \left(\frac{Ax_1 + Bx_2}{a(t)} \right) \frac{A}{a(t)}, \dot{\Phi} \left(\frac{Ax_1 + Bx_2}{a(t)} \right) \frac{B}{a(t)}, \vec{0} \right] - \frac{\alpha(N)}{a^2(t)} e^{-\frac{\Phi\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{K} + C} \quad (77)$$

$$= \frac{\partial}{\partial x_1} \left[\dot{\Phi} \left(\frac{Ax_1 + Bx_2}{a(t)} \right) \frac{A}{a(t)} \right] + \frac{\partial}{\partial x_2} \left[\dot{\Phi} \left(\frac{Ax_1 + Bx_2}{a(t)} \right) \frac{B}{a(t)} \right] - \frac{\alpha(N)}{a^2(t)} e^{-\frac{\Phi\left(\frac{Ax_1+Bx_2}{a(t)}\right)}{K} + C} \quad (78)$$

$$= \frac{A^2 + B^2}{a^2(t)} \left(\ddot{\Phi}(s) - \frac{\alpha(N)e^C}{A^2 + B^2} e^{-\frac{\Phi(s)}{K}} \right), \quad (79)$$

where we choose $s := (Ax_1 + Bx_2)/a(t)$ and the ordinary differential equation:

$$\ddot{\Phi}(s) - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \quad \Phi(0) = \alpha, \quad \dot{\Phi}(0) = \beta, \quad (80)$$

with $\frac{\alpha(N)e^C}{A^2+B^2} = \epsilon^*$, α and β are constants. Therefore, our solutions (71) satisfy the Poisson equation (4)₃.

The proof is completed. ■

Now, we are ready to check that the solutions fit into the Euler-Poisson equations (4).

Proof of Theorem 1. By Lemma 7 and Lemma 8, the solutions (61) satisfy (4)₁ and (4)₃. For

the x_1 -component of the isothermal momentum equations (4)₃ in R^N ($N \geq 3$), we have

$$\rho \left(\frac{\partial u_1}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_1}{\partial x_k} \right) + \frac{\partial}{\partial x_1} K \rho + \rho \frac{\partial \Phi}{\partial x_1} \quad (81)$$

$$= \rho \left[\frac{\partial}{\partial t} \frac{\dot{a}(t)x_1}{a(t)} + \frac{\dot{a}(t)x_1}{a(t)} \frac{\partial}{\partial x_1} \frac{\dot{a}(t)x_1}{a(t)} + \frac{\dot{a}(t)x_2}{a(t)} \frac{\partial}{\partial x_2} \frac{\dot{a}(t)x_1}{a(t)} + \sum_{k=3}^N u_k \frac{\partial}{\partial x_k} \frac{\dot{a}(t)x_1}{a(t)} \right] \quad (82)$$

$$+ K \frac{\partial}{\partial x_1} \frac{e^{-\frac{\Phi(Ax_1+Bx_2)}{K}+C}}{a^2(t)} + \rho \dot{\Phi} \left(\frac{Ax_1+Bx_2}{a(t)} \right) \frac{A}{a(t)} \quad (83)$$

$$= \rho \left[\left(\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) x_1 + \frac{\dot{a}(t)x_1}{a(t)} \frac{\dot{a}(t)}{a(t)} \right] \quad (84)$$

$$- K \frac{e^{-\frac{\Phi(Ax_1+Bx_2)}{K}+C}}{a^2(t)} \dot{\Phi} \left(\frac{Ax_1+Bx_2}{a(t)} \right) \frac{A}{Ka(t)} + \rho \dot{\Phi} \left(\frac{Ax_1+Bx_2}{a(t)} \right) \frac{A}{a(t)} \quad (85)$$

$$= \rho \left[\frac{\ddot{a}(t)x_1}{a(t)} \right] - \rho \dot{\Phi} \left(\frac{Ax_1+Bx_2}{a(t)} \right) \frac{A}{a(t)} + \rho \dot{\Phi} \left(\frac{Ax_1+Bx_2}{a(t)} \right) \frac{A}{a(t)} \quad (86)$$

$$= 0, \quad (87)$$

by taking $\ddot{a}(t) = 0$ that is

$$a(t) = a_1 + a_2 t. \quad (88)$$

For the x_2 -component of the isothermal momentum equations (4)₃ in R^N , we have

$$\rho \left(\frac{\partial u_2}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_2}{\partial x_k} \right) + \frac{\partial}{\partial x_2} K \rho + \rho \frac{\partial \Phi}{\partial x_2} \quad (89)$$

$$= \rho \left[\frac{\partial}{\partial t} \frac{\dot{a}(t)x_2}{a(t)} + \frac{\dot{a}(t)x_1}{a(t)} \frac{\partial}{\partial x_1} \frac{\dot{a}(t)x_2}{a(t)} + \frac{\dot{a}(t)x_2}{a(t)} \frac{\partial}{\partial x_2} \frac{\dot{a}(t)x_2}{a(t)} + \sum_{k=3}^N u_k \frac{\partial}{\partial x_k} \frac{\dot{a}(t)x_2}{a(t)} \right] \quad (90)$$

$$+ K \frac{\partial}{\partial x_2} \frac{e^{-\frac{\Phi(Ax_1+Bx_2)}{K}+C}}{a^2(t)} + \rho \frac{\partial \Phi \left(\frac{Ax_1+Bx_2}{a(t)} \right)}{\partial x_2} \quad (91)$$

$$= \rho \left[\left(\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) x_2 + \frac{\dot{a}(t)x_2}{a(t)} \frac{\dot{a}(t)}{a(t)} \right] \quad (92)$$

$$- K \frac{e^{-\frac{\Phi(Ax_1+Bx_2)}{K}+C}}{a^2(t)} \dot{\Phi} \left(\frac{Ax_1+Bx_2}{a(t)} \right) \frac{B}{Ka(t)} + \rho \dot{\Phi} \left(\frac{Ax_1+Bx_2}{a(t)} \right) \frac{B}{a(t)} \quad (93)$$

$$= \rho \left[\frac{\ddot{a}(t)x_2}{a(t)} \right] \quad (94)$$

$$= 0. \quad (95)$$

For the $x_i (i \geq 3)$ -component of the isothermal momentum equations $(4)_i$ in R^N , we have

$$\rho \left(\frac{\partial u_3}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_3}{\partial x_k} \right) + \frac{\partial}{\partial x_i} K \rho + \rho \frac{\partial \Phi}{\partial x_i} \quad (96)$$

$$= \rho \left[\frac{\partial}{\partial t} \frac{\dot{a}(t)x_1}{a(t)} + \frac{\dot{a}(t)x_1}{a(t)} \frac{\partial}{\partial x_1} \frac{\dot{a}(t)x_1}{a(t)} + \frac{\dot{a}(t)x_2}{a(t)} \frac{\partial}{\partial x_2} \frac{\dot{a}(t)x_1}{a(t)} + \sum_{k=1}^N u_k \frac{\partial}{\partial x_k} \frac{\dot{a}(t)x_1}{a(t)} \right] + 0 + 0 \quad (97)$$

$$= \rho \left[\left(\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) x_1 + \frac{\dot{a}(t)x_1}{a(t)} \frac{\dot{a}(t)}{a(t)} \right] \quad (98)$$

$$= \rho \frac{\ddot{a}(t)}{a(t)} x_1 \quad (99)$$

$$= 0. \quad (100)$$

Therefore, our solutions satisfy the Euler-Poisson equations. In particular, $a_1 > 0$ and $a_2 < 0$, the solutions (61) blow up in the finite time $T = -a_2/a_1$.

The proof is completed. ■

Remark 9 *The existence and uniqueness of the function $\Phi(s)$ in the solutions $(61)_3$ can be shown by the fixed point theorem, if ϵ^* is a sufficient small number.*

Additionally the blowup rate about the solutions is immediately followed:

Corollary 10 *The blowup rate of the solutions (61) is,*

$$\lim_{t \rightarrow T} \rho(t, \vec{0}) (T - t)^2 \geq O(1). \quad (101)$$

Remark 11 *The other analytical solutions in R^N can be constructed by*

$$\left\{ \begin{array}{l} \rho(t, \vec{x}) = \frac{1}{a^2(t)} e^{-\frac{\Phi(\frac{Ax_1+Bx_2}{a(t)})}{K} + C}, \quad \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{a(t)} (x_1, x_2, \tilde{x}_3, \dots, \tilde{x}_N), \\ a(t) = a_1 + a_2 t, \\ \ddot{\Phi}(s) - \epsilon^* e^{-\frac{\Phi(s)}{K}} = 0, \quad \Phi(0) = \alpha, \quad \dot{\Phi}(0) = \beta, \end{array} \right. \quad (102)$$

where $\tilde{x}_i = x_1$ or x_2 .

Remark 12 *Our solutions (7), (61) and (102) also work for the isothermal Navier-Stokes-Poisson equations in R^N ($N \geq 3$):*

$$\left\{ \begin{array}{l} \rho_t + \nabla \cdot (\rho \vec{u}) = 0, \\ (\rho \vec{u})_t + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla K \rho = -\rho \nabla \Phi + \mu \Delta \vec{u}, \\ \Delta \Phi(t, x) = \alpha(N) \rho, \end{array} \right. \quad (103)$$

where $\mu > 0$ is a positive constant.

In conclusion, due to the novel solutions obtained by the separation method, the author conjectures there exists other analytical solution in non-radial symmetry. Further works will be continued for seeking more particular solutions to understand the nature of the Euler-Poisson equations (4).

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